Lesson: Derivative Techniques -1

❖ <u>Obj</u> – The Product & Quotient Rules

After explaining to a student through various lessons and examples that:

$$\lim_{x \to 8} \frac{1}{x-8} = \infty$$

I tried to check if she really understood that, so I gave her a different example. This was the result:

$$\lim_{x \to 5} \frac{1}{x-5} = 5$$

❖ <u>When the Power Rule Fails</u>:

<u>Example</u>: Differentiate

$$\frac{dy}{dx}\left[\frac{x^3+2x^2-1}{x+5}\right]$$

❖ The problem with this situation is that the function we are trying to differentiate is a combination of two functions.

❖ For situations like this, there are two techniques that can be used to differentiate the function.

(i) The Product Rule
(ii) The Quotient Rule

3.4.1 THEOREM (*The Product Rule*). *If f and g are differentiable at x, then so is the product $f \cdot g$, and*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] \quad (1)$$

In words, *the derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.*

- An easier to remember version of the Product Rule is:

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f'\cdot g + f\cdot g'$$

Some, however, prefer to write it as:

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = g\cdot f' + f\cdot g'$$

❖ Proof of Product Rule

$$\frac{d}{dx}[f(x)\cdot g(x)] = f'\cdot g + f\cdot g'$$

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right]$$

$$= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x+h) \cdot g(x) + f(x+h) \cdot g(x) - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} \left[f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + g(x) \cdot \frac{f(x+h) - f(x)}{h} \right]$$

$$= \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} g(x) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} f(x+h) \cdot g'(x) + \lim_{h \to 0} g(x) \cdot f'(x)$$

$$= \lim_{h \to 0} f(x+h) \cdot g'(x) + \lim_{h \to 0} g(x) \cdot f'(x)$$

$$= f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

$$= f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Q.E.D.

Quod Erat Demonstrandum,

Which means “Thus it is demonstrated”.

Ex. 1: Find $\frac{dy}{dx}$ if

$$y = (4x^2 - 1)(7x^3 + x)$$

Method 1: FOIL & Differentiate

Method 2: "Product Rule"

Ex. 2: Find y' if

$$y = (3x^2 + 2x - 5)(6x^2 + 7)$$

Ex. 3: Differentiate

$$f(x) = (1 + x) \cdot \sqrt{x}$$

3.4.2 THEOREM (*The Quotient Rule*). *If f and g are differentiable at x and $g(x) \neq 0$, then f/g is differentiable at x and*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} \qquad (2)$$

In words, *the derivative of a quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared.*

➢An easier to remember version of the Quotient Rule is:

$$\frac{dy}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f' \cdot g - f \cdot g'}{g^2}$$

<u>Proof of Quotient Rule</u>:

$$\frac{dy}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{g^2}$$

$$\frac{dy}{dx}\left[\frac{f(x)}{g(x)}\right]=\lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)\cdot g(x)-f(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)}$$

$$=\lim_{h\to 0}\frac{f(x+h)\cdot g(x)-f(x)\cdot g(x)-f(x)\cdot g(x+h)+f(x)\cdot g(x)}{h\cdot g(x)\cdot g(x+h)}$$

$$=\lim_{h\to 0}\frac{g(x)\cdot[f(x+h)-f(x)]-f(x)\cdot[g(x+h)-g(x)]}{h\cdot g(x)\cdot g(x+h)}$$

$$=\frac{\lim_{h\to 0}g(x)\cdot\lim_{h\to 0}[f(x+h)-f(x)]-\lim_{h\to 0}f(x)\cdot\lim_{h\to 0}[g(x+h)-g(x)]}{h\cdot g(x)\cdot g(x+h)}$$

$$=\frac{\frac{\lim_{h\to 0}g(x)\cdot\lim_{h\to 0}[f(x+h)-f(x)]}{h}-\frac{\lim_{h\to 0}f(x)\cdot\lim_{h\to 0}[g(x+h)-g(x)]}{h}}{g(x)\cdot g(x+h)}$$

$$=\frac{\lim_{h\to 0}g(x)\cdot\frac{\lim_{h\to 0}[f(x+h)-f(x)]}{h}-\lim_{h\to 0}f(x)\cdot\frac{\lim_{h\to 0}[g(x+h)-g(x)]}{h}}{g(x)\cdot g(x+h)}$$

$$= \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x) \cdot g(x+0)}$$

$$= \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x) \cdot g(x)}$$

$$= \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

$$= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

Q.E.D.

<u>Ex. 4</u>: Differentiate

$$y = \left[\frac{x^3 + 2x^2 - 1}{x + 5} \right]$$

f (numerator) ← ; g (denominator) ←

$$\frac{dy}{dx}\left[\frac{f(x)}{g(x)} \right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2}$$

Ex. 5: Find

(a) $\frac{dy}{dx}\left[\frac{x-1}{x^2+3}\right]$

(b) Where does the function above have horizontal tangent lines?

Ex. 6: For $y = \dfrac{3x^2 - 5x + 2}{2x + 7}$, Find

$$\left.\frac{dy}{dx}\right|_{x=1}$$

Ex. 7:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
3	5	- 3	- 4	7

The table above gives values of the differentiable functions f and g and their derivatives at x = 3.

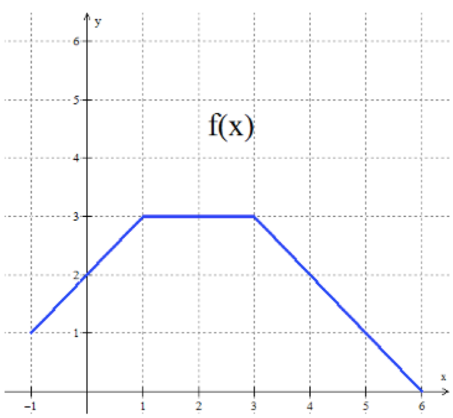
If h(x) = (3 f(x) + 2)(5 - g(x)), then h'(3) =

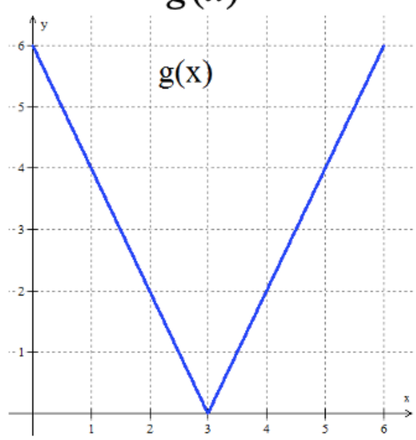
Ex. 8: find dy/dx, where d is a constant.

$$y = \frac{x^2 + d^2}{x^2 - d^2}$$

Given the graphs of f and g below: If

$$r(x) = f(x) \cdot g(x) \text{ and } w(x) = \frac{f(x)}{g(x)}$$



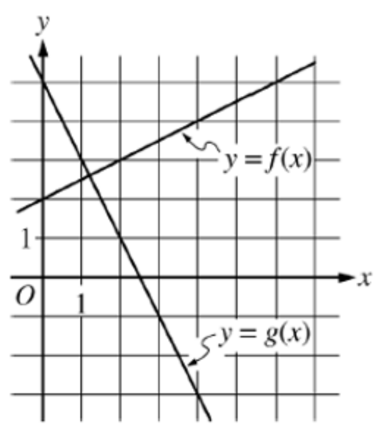


$r'(2) = ?$ $w'(4) = ?$ $w'(1) = ?$

AP Practice

If $y = \frac{2x+3}{3x+2}$, then $\frac{dy}{dx} =$

(A) $\frac{12x+13}{(3x+2)^2}$ (B) $\frac{12x-13}{(3x+2)^2}$ (C) $\frac{5}{(3x+2)^2}$ (D) $\frac{-5}{(3x+2)^2}$ (E) $\frac{2}{3}$



The figure above shows the graphs of the functions f and g. If $h(x) = f(x)g(x)$, then $h'(2) =$

(A) $\frac{13}{2}$ (B) $\frac{1}{2}$ (C) -1 (D) $-\frac{11}{2}$